Fourier Analysis 04-02-2024.

Review.

Consider the heat equation on the real line

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, & x \in \mathbb{R},\ t > 0 \\ u(x,0) = f(x). \end{cases}$$

Using the method of Fourier transform, we have derived a formal solution $u(x,t)$ as follows:

$$u(x,t) = f * \mathcal{H}_t(x),$$

where

$$\mathcal{H}_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}, \qquad x \in \mathbb{R},\ t > 0.$$

We call $\{\mathcal{H}_t\}_{t>0}$ the heat kernel on $\mathbb{R}$.

Below we will provide a theoretic check.

Schwartz space:

Let $S(\mathbb{R})$ denote the collection of all C^∞ functions f on $\mathbb{R}$ such that for any $k, \ell \geq 0$,

$$\sup_{x \in \mathbb{R}} |x|^k \cdot |f^{(\ell)}(x)| < \infty.$$

Here we collect some easily checked properties of $S(\mathbb{R})$.

- $S(\mathbb{R})$ is a vector space over $\mathbb{C}$
- $\forall f \in S(\mathbb{R})$, $\quad f' \in S(\mathbb{R})$.
- $\forall f \in S(\mathbb{R})$, $\quad x f(x) \in S(\mathbb{R})$.

  More general, $\quad p(x) f(x) \in S(\mathbb{R})$ for each polynomial p.

Prop 2: $f \in S(\mathbb{R}) \Leftrightarrow \hat{f} \in S(\mathbb{R})$.

Pf. We only prove the direction:

$$f \in S(\mathbb{R}) \Rightarrow \hat{f} \in S(\mathbb{R}).$$

The other direction follows from the Fourier inversion formula.

Suppose $f \in S(\mathbb{R})$.

We need to show that

$$\sup_{\xi \in \mathbb{R}} |\xi|^k \cdot \left| \hat{f}^{(\ell)}(\xi) \right| < \infty \quad \forall \ k, \ell \geq 0.$$

Notice that

$$(-2\pi i x)^\ell f(x) \xrightarrow{\mathcal{F}} \hat{f}^{(\ell)}(\xi)$$

$$F(x) := \frac{d^k\left(\left(-2\pi i x\right)^\ell f(x)\right)}{dx^k} \xrightarrow{\mathcal{F}} \left(2\pi i \xi\right)^k \cdot \hat{f}^{(\ell)}(\xi)$$

In particular

$$\sup_{\xi \in \mathbb{R}} \left| \left(2\pi i \xi\right)^k \hat{f}^{(\ell)}(\xi) \right| \leq \int_{-\infty}^{\infty} \left| F(x) \right| dx$$

$$< \infty \quad \left(\text{since } F \in \mathcal{S}(\mathbb{R})\right)$$

$$\Rightarrow \quad \sup_{\xi \in \mathbb{R}} |\xi|^k \cdot \left|\hat{f}^{(\ell)}(\xi)\right| < \infty .$$

$\boxed{/\!/}$

Thm 3. Let $f \in S(\mathbb{R})$. Let

$$u(x,t) = f * \mathcal{H}_t(x).$$

Then

① $u \in C^\infty(\mathbb{R} \times \mathbb{R}_+)$, $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ on $\mathbb{R} \times \mathbb{R}_+$

② $u(x,t) \rightrightarrows f(x)$ as $t \to 0$

③ $\displaystyle\int_{\mathbb{R}} |u(x,t) - f(x)|^2 dx \to 0$ as $t \to 0$.

Pf. Since $u = f * \mathcal{H}_t$, both $f, \mathcal{H}_t \in S(\mathbb{R})$,

it is not hard to show that

$$f * \mathcal{H}_t \in S(\mathbb{R}) \quad \forall\, t > 0.$$

Now by Fourier inversion formula,

$$u(x,t) = \int_{-\infty}^{\infty} \hat{u}(\xi,t)\, e^{2\pi i \xi x} dx$$

(*) $$= \int_{-\infty}^{\infty} \hat{f}(\xi) \cdot e^{-4\pi^2\xi^2 t} e^{2\pi i \xi x} \, d\xi$$

Let us show

$\frac{\partial u}{\partial t}$ exists

$$\frac{u(x, t+\Delta t) - u(x,t)}{\Delta t} = \int_{-\infty}^{\infty} \hat{f}(\xi) \cdot \frac{e^{-4\pi^2\xi^2(t+\Delta t)} - e^{-4\pi^2\xi^2 t}}{\Delta t} \cdot e^{2\pi i \xi x} \, d\xi$$

$$= \int_{-\infty}^{\infty} \hat{f}(\xi) \cdot e^{-4\pi^2\xi^2 t} \cdot \frac{e^{-4\pi^2\xi^2\Delta t} - 1}{\Delta t} e^{2\pi i \xi x} \, d\xi$$

Notice that

$$\left| \frac{e^{-4\pi^2\xi^2\Delta t} - 1}{\Delta t} \right| \leq \text{Const} \cdot |\xi^2|$$

$$\lim_{\Delta t \to 0} \frac{e^{-4\pi^2\xi^2\Delta t} - 1}{\Delta t} = (-4\pi^2\xi^2)$$

By Lebesgue's dominated convergence Thm,

$$\lim_{\Delta t \to 0} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{-4\pi^2\xi^2 t} \frac{e^{-4\pi^2\xi^2\Delta t} - 1}{\Delta t} \cdot e^{2\pi i \xi x} d\xi$$

$$= \int_{-\infty}^{\infty} \hat{f}(\xi) e^{-4\pi^2\xi^2 t} \cdot (-4\pi^2\xi^2) \cdot e^{2\pi i \xi x} d\xi$$

By similarly arguments, we see that

$$u \in C^\infty(\mathbb{R} \times \mathbb{R}_+).$$

By (*), we have $\frac{\partial^2 u}{\partial x^2} = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{-4\pi^2\xi^2 t} \cdot e^{2\pi i \xi x} (2\pi i \xi)^2 d\xi$

and $\frac{\partial u}{\partial t} = \int_{-\infty}^{\infty} \hat{f}(\xi) \cdot e^{-4\pi^2\xi^2 t} (-4\pi^2\xi^2) e^{2\pi i \xi x} d\xi$.

So $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ on $\mathbb{R} \times \mathbb{R}_+$. This proves ①.

Part ② is a consequence that

$$\{\mathcal{H}_t\}_{t>0}$$

is a good kernel on $\mathbb{R}$.

Below we prove ③:

By Plancherel formula,

$$\int_{-\infty}^{\infty} |u(x,t)-f(x)|^2\,dx$$

$$=\int_{-\infty}^{\infty} |\hat{u}(\xi,t)-\hat{f}(\xi)|^2\,d\xi$$

$$=\int_{-\infty}^{\infty} |\hat{f}(\xi)\cdot e^{-4\pi^2\xi^2 t}-\hat{f}(\xi)|^2\,d\xi$$

$$=\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2\cdot|e^{-4\pi^2\xi^2 t}-1|^2\,d\xi$$

Notice that

$$|\hat{f}(\xi)|^2\cdot|e^{-4\pi^2\xi^2 t}-1|^2 \leq |\hat{f}(\xi)|^2$$

By DCT,

$$\lim_{t \to 0} \int |\hat{f}(\xi)|^2 \left| e^{-4\pi^2 \xi^2 t} - 1 \right|^2 d\xi$$

$$= \int_{\mathbb{R}} \lim_{t \to 0} |\hat{f}(\xi)|^2 \cdot \left| e^{-4\pi^2 \xi^2 t} - 1 \right|^2 d\xi$$

$$= 0.$$

$\blacksquare$

Q: Is $u = f * \mathcal{H}_t$ the unique solution to the heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} & \text{on } \mathbb{R} \times \mathbb{R}_+, \\ u(x,0) = f(x). \end{cases}$$

The answer is no!

Example 1. Let $u(x,t) = \dfrac{x}{t} \mathcal{H}_t(x)$ on $\mathbb{R} \times \mathbb{R}_+$.

Check: • $u \in C^\infty(\mathbb{R} \times \mathbb{R}_+)$

$$(*) \begin{cases} \bullet\ \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} & \text{on } \mathbb{R} \times \mathbb{R}_+ \\ \bullet\ \lim\limits_{t \to 0} u(x,t) = 0. \end{cases}$$

However $u \equiv 0$ is also a solution of $(*)$.

So $(*)$ has more than one solution.